NORTHEASTERN UNIVERSITY
DEPARTMENT OF PHYSICS

Final Exam

Student name: *Nicholas Hurley*

Course: *Quantum Mechanics I (PHYS4115)* – Professor: *Prof. Dima Krioukov*
Due date: *December 14th, 2020*

All computations were made in Mathematica. The Mathematica code is attached to the end of the document as an Appendix.

Question 1

Observe that $j = \hat{R}_{SO(3)}(\frac{-\pi}{2}, i)k$ so that the eigenstates of the operator $\hat{S}_y$ for a spin-1 particle are given by $|1, m\rangle_y = \hat{R}_{SU(2)}(\frac{-\pi}{2}, i)\,|1, m\rangle_z\,, m = 1, 0, -1$ Find their explicit vector representations following the methodology used in the class to derive those for $|1, m\rangle_x$.

Answer. We first observe that $j = \hat{R}_{SO(3)}(\frac{-\pi}{2}, i)k$. We can therefore conclude that the eigenstates of $\hat{S}_y$ for a spin-1 particle are given by $|1, m\rangle_y = \hat{R}_{SU(2)}(\frac{-\pi}{2}, i)\,|1, m\rangle_z\,, m = 1, 0, -1$.

We can define $\hat{R}_{SU(2)}(\frac{-\pi}{2}, i)$ in terms of the generator $\hat{S}_x$ with $\hat{R}_{SU(2)}(\frac{-\pi}{2}, i) = e^{\frac{-i}{\hbar}\hat{S}_x\frac{-\pi}{2}}$. From class, we know $\hat{S}_x$ to be:

$$\begin{pmatrix} 0 & \frac{\hbar}{\sqrt{2}} & 0 \\ \frac{\hbar}{\sqrt{2}} & 0 & \frac{\hbar}{\sqrt{2}} \\ 0 & \frac{\hbar}{\sqrt{2}} & 0 \end{pmatrix}$$

Using matrix exponents in Mathematica, we then obtain:

$$\hat{R}_{SU(2)}(\frac{-\pi}{2}, i) = e^{\frac{-i}{\hbar}\hat{S}_x\frac{-\pi}{2}}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{i}{\sqrt{2}} & -\frac{1}{2} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ -\frac{1}{2} & \frac{i}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

Taking the columns of this matrix, we can then find the eigenstates of $\hat{S}_y$. Therefore:

$$|1, 1\rangle_y = \begin{pmatrix} \frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ -\frac{1}{2} \end{pmatrix}$$

$$|1,0\rangle_y = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{pmatrix}$$

$$|1,-1\rangle_y = \begin{pmatrix} -\frac{1}{2} \\ \frac{i}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

Question 2

The correct solutions to the last problem (Problem 6) of the Midterm Exam about the spin projection operators of the graviton, a spin-2 particle, are:

$$\hat{S}_x = \begin{pmatrix} 0 & \hbar & 0 & 0 & 0 \\ \hbar & 0 & \sqrt{\frac{3}{2}}\hbar & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}}\hbar & 0 & \sqrt{\frac{3}{2}}\hbar & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}}\hbar & 0 & \hbar \\ 0 & 0 & 0 & \hbar & 0 \end{pmatrix}$$

$$\hat{S}_y = \begin{pmatrix} 0 & -i\hbar & 0 & 0 & 0 \\ i\hbar & 0 & -i\sqrt{\frac{3}{2}}\hbar & 0 & 0 \\ 0 & i\sqrt{\frac{3}{2}}\hbar & 0 & -i\sqrt{\frac{3}{2}}\hbar & 0 \\ 0 & 0 & i\sqrt{\frac{3}{2}}\hbar & 0 & -i\hbar \\ 0 & 0 & 0 & i\hbar & 0 \end{pmatrix}$$

Find the explicit 5 x 5-matrix representations of rotation operators:

$$\hat{R}_x = \hat{R}_{SU(2)}(\frac{\pi}{2}, j) = e^{\frac{-i}{\hbar}\hat{S}_y\frac{\pi}{2}}$$

$$\hat{R}_y = \hat{R}_{SU(2)}(\frac{-\pi}{2}, i) = e^{\frac{-i}{\hbar}\hat{S}_x\frac{-\pi}{2}}$$

and verify by explicit calculations that the columns of the $\hat{R}_x$ and $\hat{R}_y$ matrices are the eigenstates $|2,m\rangle_x$ and $|2,m\rangle_y$ of the operators $\hat{S}_x$ and $\hat{S}_y$, respectively, where m = +2, +1, 0, -1, -1.

Answer. We first use matrix exponentiation in Mathematica to find the matrices $\hat{R}_x$ and $\hat{R}_y$ using the equations given and the matrices $\hat{S}_x$ and $\hat{S}_y$. We will then confirm that the columns of these matrices are in fact eigenstates by multiplying them by $\hat{S}_x$ and $\hat{S}_y$ respectively. Each column m of the product matrix will correspond to the product vector $\hat{S}_a |2,m\rangle_a = m\hbar |2,m\rangle_a$ for $a = x, y$. Doing these calculations in Mathematica for a = x, we obtain:

$$\hat{R}_x = e^{\frac{-i}{\hbar}\hat{S}_y\frac{\pi}{2}} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & -\frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{\frac{3}{2}}}{2} & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$\hat{S}_x\hat{R}_x = \begin{pmatrix} \frac{\hbar}{2} & -\frac{\hbar}{2} & 0 & \frac{\hbar}{2} & -\frac{\hbar}{2} \\ \hbar & -\frac{\hbar}{2} & 0 & -\frac{\hbar}{2} & \hbar \\ \sqrt{\frac{3}{2}}\hbar & 0 & 0 & 0 & -\sqrt{\frac{3}{2}}\hbar \\ \hbar & \frac{\hbar}{2} & 0 & \frac{\hbar}{2} & \hbar \\ \frac{\hbar}{2} & \frac{\hbar}{2} & 0 & -\frac{\hbar}{2} & -\frac{\hbar}{2} \end{pmatrix}$$

From left to right, each column vector of $\hat{R}_x$ corresponds to $|2, m\rangle_x$ for m = 2, 1, 0, -1, -2. Looking at the columns of $\hat{S}_x\hat{R}_x$, one finds they take the form $m\hbar\,|2, m\rangle_x$, confirming that each column vector of $\hat{R}_x$ is indeed an eigenstate of $\hat{S}_x$

We then do the same thing for a = y.

$$\hat{R}_y = e^{\frac{-i}{\hbar}\hat{S}_x\frac{-\pi}{2}} = \begin{pmatrix} \frac{1}{4} & \frac{i}{2} & -\frac{\sqrt{\frac{3}{2}}}{2} & -\frac{i}{2} & \frac{1}{4} \\ \frac{i}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{i}{2} \\ -\frac{\sqrt{\frac{3}{2}}}{2} & 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{\frac{3}{2}}}{2} \\ -\frac{i}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{i}{2} \\ \frac{1}{4} & -\frac{i}{2} & -\frac{\sqrt{\frac{3}{2}}}{2} & \frac{i}{2} & \frac{1}{4} \end{pmatrix}$$

$$\hat{S}_y\hat{R}_y = \begin{pmatrix} \frac{\hbar}{2} & \frac{i\hbar}{2} & 0 & \frac{i\hbar}{2} & -\frac{\hbar}{2} \\ i\hbar & -\frac{\hbar}{2} & 0 & \frac{\hbar}{2} & i\hbar \\ -\sqrt{\frac{3}{2}}\hbar & 0 & 0 & 0 & \sqrt{\frac{3}{2}}\hbar \\ -i\hbar & -\frac{\hbar}{2} & 0 & \frac{\hbar}{2} & -i\hbar \\ \frac{\hbar}{2} & -\frac{1}{2}(i\hbar) & 0 & -\frac{1}{2}(i\hbar) & -\frac{\hbar}{2} \end{pmatrix}$$

From left to right, each of the column vectors in the product matrix takes on a value of $\hbar m\,|2, m\rangle_y$ for m = 2, 1, 0, -1, -2.

Question 3

A graviton is in the spin state $c\begin{pmatrix} -2+2i \\ -1+i \\ 1+i \\ 1-i \\ 2-2i \end{pmatrix}$, where c is a normalization constant.
Determine all the 3 x 5 = 15 probabilities P_{am} that the measurement of the spin projection of the graviton along the axis a = x, y, z will yield value m, where m = +2, +1, 0, -1, -2. (The solutions in this problem should be written down in the fully simplified algebraic form. That is, expressions like "2/5" and "$2\sqrt{5}$" would be fine, but points will be taken if they are written instead as "0.4" (or "4/10") and "4.47" (or "$\sqrt{20}$").)

Answer. We first normalize $|\psi\rangle$.

$$\langle\psi|\,|\psi\rangle = 1$$

$$= c^2(|-2+2i|^2+|-1+i|^2+|1+i|^2+|1-i|^2+|2-2i|^2) = c^2(8+2+2+2+8) = 22c^2$$

$$c^2 = \frac{1}{22}$$

$$c = \frac{1}{\sqrt{22}}$$

Therefore

$$|\psi\rangle = \frac{1}{\sqrt{22}}\begin{pmatrix} -2+2i \\ -1+i \\ 1+i \\ 1-i \\ 2-2i \end{pmatrix}$$

The probability P_{am} for finding $|\psi\rangle$ in each eigenstate can be found by taking $|\,\langle 2,m|_a\,|\psi\rangle\,|^2$ for m = +2, +1, 0, -1, -2 and for a = x, y, z. We use the columns of $\hat{R}_x$ and $\hat{R}_y$ from Problem 2 as the eigenstates of $\hat{S}_x$ and $\hat{S}_x$ respectively, and the columns of the identity matrix to represent the eigenstates of $\hat{S}_z$.

The coefficients $\langle\psi|\,|2,m\rangle_a$ for each row a and column m can be represented with the matrix:

$$\begin{pmatrix} \langle\psi|\,\hat{R}_x \\ \langle\psi|\,\hat{R}_y \\ \langle\psi|\,\hat{I} \end{pmatrix}$$

When we multiply each element of this matrix by its complex conjugate in Mathematica, we obtain:

$$P_{am} = \begin{pmatrix} \frac{3}{88} & \frac{1}{11} & \frac{1}{44} & \frac{9}{11} & \frac{3}{88} \\ \frac{1}{8} - \frac{\sqrt{\frac{3}{2}}}{11} & \frac{4}{11} & \frac{1}{44} & \frac{4}{11} & \frac{\sqrt{\frac{3}{2}}}{11} + \frac{1}{8} \\ \frac{4}{11} & \frac{1}{11} & \frac{1}{11} & \frac{1}{11} & \frac{4}{11} \end{pmatrix}$$

The rows take on the values a = x, y, and z from top to bottom. The columns take on the values m = +2, +1, 0, -1, -2 from left to right. For example, P_{x1} is the probability of measuring $|\psi\rangle$ with a spin-1 projection on the x-axis. This probability is found in the first row and the second column in the matrix, with a value of $\frac{1}{11}$.

Notice that the probabilities in each row add up to 1, confirming that $|\psi\rangle$ must be found in one of five eigenstates when measured with respect to directions x, y, or z.

Question 4)

At time t = 0, a spinless one-dimensional free particle of mass t is in the state

$$\psi(p,0) = \langle p| |\psi(0)\rangle = \sqrt{\frac{a}{\hbar\sqrt{\pi}}} e^{-\frac{a^2(p-p_0)^2}{2\hbar^2}} e^{-\frac{ix_0(p-p_0)}{\hbar}}$$

where a > 0 and x_0, $p_0 \in \mathbb{R}$. This state is the minimum uncertainty state with <x> = x_0 and <p> = p_0.

Answer.

a. $P(p,0)$

This can be found by taking $\psi(p,0)\psi^*(p,0)$. This yields the result:

$$P(p,0) = \frac{a}{\sqrt{\pi}\hbar} e^{-\frac{a^2(p-\mathrm{p0})^2}{\hbar^2}}$$

This state is a Gaussian similar to the one described in class, with a mean of p0.

b. $\psi(x,0)$

This can be found by taking the inverse Fourier transform of $\psi(p,0)$.

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}px} \psi(p,0) dp$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}px} \psi(p,0) dp$$
$$= \sqrt{\frac{a}{2\pi\sqrt{\pi}\hbar^2}} \int_{-\infty}^{\infty} dp e^{\frac{i}{\hbar}px} e^{-\frac{a^2(p-p_0)^2}{2\hbar^2}} e^{-\frac{ix_0(p-p_0)}{\hbar}}$$

Taking this integral in Mathematica, one obtains the result:

$$\psi(x,0) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-\frac{(x-\mathrm{x0})^2}{2a^2}} e^{\frac{i\mathrm{p0}x}{\hbar}}$$

c. $P(x,0)$

This can be found by taking $\psi(x,0)\psi^*(x,0)$. This means that the normalization constant is squared and the phase term is removed.

$$P(x,0) = \psi(x,0)\psi^*(x,0) = \frac{1}{a\sqrt{\pi}} e^{-\frac{(x-\mathrm{x0})^2}{a^2}}$$

.

This is a Gaussian with a mean of x0.

d. $\psi(p,t)$

Since the particle is free, $\psi(p,0)\,|p\rangle$ is an eigenstate of the Hamiltonian. $\psi(p,0)$ can be evolved by multiplying by $e^{\frac{-i}{\hbar}Et}$, with $E=\frac{p^2}{2m}$. Therefore:

$$\psi(p,t)=e^{\frac{-ip^2t}{2m\hbar}}\psi(p,0)=\sqrt{\frac{a}{\hbar\sqrt{\pi}}}e^{\frac{-ip^2t}{2m\hbar}}e^{-\frac{a^2(p-p_0)^2}{2\hbar^2}}e^{-\frac{ix_0(p-p_0)}{\hbar}}$$

e. $P(p,t)$

This can again be found by taking $\psi(p,0)\psi^*(p,0)$. Since time evolution only adds a phase shift to each eigenbasis in the momentum-space, the probability density in the momentum space is preserved, and therefore:

$$P(p,t)=P(p,0)=\frac{a}{\sqrt{\pi}\hbar}e^{-\frac{a^2(p-\mathrm{p0})^2}{\hbar^2}}$$

f. $\psi(x,t)$

This can be found by again taking the inverse Fourier transform of $\psi(p,t)$.

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{\frac{i}{\hbar}px}\psi(p,t)dp$$

In Mathematica, one obtains the result:

$$\psi(x,t)=\frac{\exp\left(\frac{a^2\mathrm{p0}(2mx-\mathrm{p0}t)+i\hbar\left(m(x-\mathrm{x0})^2+2\mathrm{p0}t\mathrm{x0}\right)}{2\hbar(t\hbar-ia^2m)}\right)}{\sqrt[4]{\pi}\sqrt{a+\frac{it\hbar}{am}}}$$

This can be further simplified to:

$$\psi(x,t)=\frac{1}{\sqrt{\sqrt{\pi}a(1+\frac{it\hbar}{ma^2})}}e^{\frac{-(x-(x_0+p_0t/m))^2}{2a^2(1+(\frac{\hbar t}{ma^2})^2)}}e^{i\frac{a^4m\mathrm{p0}(2mx-\mathrm{p0}t)+t\hbar^2\left(m(x-\mathrm{x0})^2+2\mathrm{p0}t\mathrm{x0}\right)}{2\left(a^4m^2\hbar+t^2\hbar^3\right)}}$$

g. $P(x,t)$

This can be found by taking $\psi(x,t)\psi^*(x,t)$. In Mathematica, one obtains:

$$P(x,t)=\frac{am\exp\left(-\frac{a^2(m(\mathrm{x0}-x)+\mathrm{p0}t)^2}{a^4m^2+t^2\hbar^2}\right)}{\sqrt{\pi}\sqrt{a^4m^2+t^2\hbar^2}}$$

Simplified, this takes the form:

$$P(x,t)=\frac{1}{a\sqrt{\pi(1+(\frac{t\hbar}{ma^2})^2)}}e^{\frac{-(x-(x_0+p_0t/m))^2}{a^2(1+(\frac{\hbar t}{ma^2})^2)}}$$

This is a Gaussian with a mean of $x_0 + p_0 t/m$. Notice how the mean corresponds to the initial displacement of x_0 plus the displacement due to some average momentum p_0 over some time t. The standard deviation is $\frac{a}{\sqrt{2}}\sqrt{1+(\frac{\hbar t}{ma^2})^2}$. This means that the Gaussian widens with time.

h. <x>(t)

We take $\int_{-\infty}^{\infty} xP(x,t)dx$. This yields:

$$< x > (t) = (p0t)/m + x0$$

This result is the classical limit of a free particle with momentum p0. If a classical particle has velocity $\frac{p0}{m}$ and initial position x0, after time t it will be at position $\frac{p0t}{m} + x0$. This result again shows that expectation values of quantum states approach classical approximations.

i. $< p > (t)$

This can be found in a similar manner to $< x > (t)$.

$$< p > (t) = \int_{-\infty}^{\infty} pP(p,t)dp = p0$$

Since the momentum-probability distribution is a Gaussian with a mean of p0, p0 should be expected as an average value of the momentum. In the classical limit, this is a particle with a constant velocity. The average value of p does not change with time since the momentum operator commutes with the Hamiltonian.

j. ΔX

This can be found with:

$$\Delta X = \sqrt{< x^2 > - < x >^2}$$

$< x^2 >$ can be found with $\int_{-\infty}^{\infty} x^2 P(x,t)dx$.

In Mathematica, one obtains the result:

$$\begin{aligned} < x^2 > &= \frac{a^2}{2} + ((p0t)/m + x0)^2 + \frac{t^2\hbar^2}{2ma^2} \\ &= \frac{a^2}{2} + < x >^2 + \frac{t^2\hbar^2}{2ma^2} \end{aligned}$$

$$\begin{aligned} \Delta X = \sqrt{< x^2 > - < x >^2} &= \sqrt{\frac{a^2}{2} + \frac{t^2\hbar^2}{2ma^2}} \\ &= \frac{a}{\sqrt{2}}\sqrt{1+(\frac{t\hbar}{a^2 m})^2} \end{aligned}$$

The uncertainty of x begins at $\frac{a}{\sqrt{2}}$ but grows with time, meaning that the Gaussian corresponding to P(x, 0) becomes wider with time. This result is very similar to the form found for the Gaussian distribution discussed in class. This means that the position-space probability distribution grows wider with time, and its peak translates through space with time by a factor of p_0/m.

Something interesting happens when we take the variance of x.

$$\Delta X^2 = \frac{a^2}{2} + (\frac{t\hbar}{\sqrt{2}am})^2$$

Using the result from part k that $\Delta p(t) = \frac{\hbar}{\sqrt{2}a}$, we obtain:

$$\Delta X^2 = \frac{a^2}{2} + (\frac{\Delta pt}{m})^2$$

Similar to with the average value of x, this result shows that the variance of x with time is equal to some initial variance $\frac{a^2}{2}$ plus some displacement due to uncertainty in the momentum. With greater uncertainty in momentum, the range of possible velocities of the particle is greater, and thus the range of possible displacements of the particle spreads with time.

k. Δp.

This can be found using identical methods as with x.

$< p^2 >$ can be found with $\int_{-\infty}^{\infty} p^2 P(p,t)dp$.

$$< p^2 >= \frac{\hbar^2}{2a^2} + \mathrm{p0}^2$$

$$\Delta p = \sqrt{< p^2 > - < p >^2} = \sqrt{\frac{\hbar^2}{2a^2} + \mathrm{p0}^2 - \mathrm{p0}^2}$$
$$= \frac{\hbar}{\sqrt{2}a}$$

The uncertainty in p does not change with time, which is to be expected since P(p, t) is constant with time.

l. $\Delta x \Delta p$

$$\Delta x \Delta p(t) = \frac{\hbar}{\sqrt{2}a}\frac{a}{\sqrt{2}}\sqrt{1 + (\frac{t\hbar}{a^2m})^2}$$
$$= \frac{\hbar}{2}\sqrt{1 + (\frac{t\hbar}{a^2m})^2}$$

Notice how at t = 0, $\Delta x \Delta p$ is at the minimum uncertainty of $\frac{\hbar}{2}$. However, with time, the probability distribution spreads in the position space, and therefore the uncertainty product increases.

Problem 5

A spinless one-dimensional particle whose mass is the same as the mass of the electron is in a potential well of height V_0 = 2 eV: V(x) = 0 if |x| < w/2, and V(x) = V_0 otherwise. What is the minimum width w of the well in angstroms, so that the ground state energy does not exceed 1eV?

Answer. We shall solve for the Schrodinger equation within each region.

$$|x| > w/2$$

In this region, $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = (E - V_0)\psi(x)$. E - V_0 < 0, therefore:

$$\psi(x) = Ae^{qx} + Be^{-qx}$$

where $q = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$. In order to satisfy normalization, the function must not go to infinity for high and low x. Therefore:

$$\psi(x) = \begin{cases} Ae^{-qx} & x > w/2 \\ Be^{qx} & x < -w/2 \end{cases}$$

We now consider the region for which $|x| < w/2$. In this region, $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x)$. Therefore:

$$\psi(x) = A\cos(kx) + B\sin(kx)$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$. This solution has a symmetric case where B = 0, and an anti-symmetric case where A = 0.

These piece-wise solutions must be continuous and have the same derivative at the boundaries $x = \pm w/2$. Let's satisfy the boundary condition at $x = w/2$ for the symmetric case.

$$\psi(w/2) = Be^{-qw/2} = A\cos(kw/2)$$

$$\psi'(w/2) = -qBe^{-qw/2} = -kA\sin(kw/2)$$

Combining these two equations, we get:

$$q = k\tan(kw/2)$$

Repeating the same process for the anti-symmetric case, we get:

$$q = -k\cot(kw/2)$$

Plugging in the values V_0 = 2eV, E = 1eV,and m = .51 Mev$/c^2$, we can find numerical values for k and q. When we solve the two equations for w in Mathematica, we get the following results.

$$w = \begin{cases} 3.07038 + 12.2815\text{n} & \text{angstroms, symmetric case} \\ -3.07038 + 12.2815\text{n} & \text{angstroms, antisymmetric case} \end{cases}$$

As can be seen, the smallest non-negative value for w is found when the wavefunction is symmetric and n = 0, since n belongs to the integers. This yields the solution:

w = 3.07038 angstroms

Problem 6

The position-space wave function of a free spinless one-dimensional particle whose mass is the same as the mass of the electron, is initially constant in a one-meter-long interval, and zero everywhere else, so that the particle is initially in this interval for sure.

a) What is the probability that the particle is still in this interval in one second? One minute? One hour? One day? One week? One month? One year?

b) Do your best describing how this probability changes with time when time is very large.

c) Do your best describing and visualizing how the overall position probability distribution evolves with time when time changes from zero to infinity.

Answer.

a. We first define the wave-function. The wave-function is constant at one such that it is properly normalized over a 1 meter interval.

$$\psi(x,0) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \text{ m} \\ 1 & 0 \leq x \leq 1 \text{ m} \end{cases}$$

We shall convert this function to the momentum-space by taking the Fourier transform. The integral only needs to be defined on the interval $0 \leq x \leq 1$ since it is zero otherwise.

$$\psi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^1 e^{-\frac{i}{\hbar}px} 1 dx$$

We can then evolve this state with time with the operator

$$\hat{U}(t) = e^{-\frac{i}{\hbar}\hat{H}t}$$

Since the particle is free, the potential energy, and as such:

$$\hat{H} = \frac{\hat{p}^2}{2m}$$

Each momentum eigenstate is therefore also an eigenstate of $\hat{H}$ with eigenvalue $\frac{p^2}{2m}$. Therefore:

$$\hat{U}(t)\,|p\rangle = e^{-\frac{ip^2}{2m\hbar}t}\,|p\rangle$$

We can therefore evolve $|\psi\rangle$ with time by evolving each of its basis vectors in the position space.

$$|\psi,t\rangle = \int_{-\infty}^{-\infty} dp'\hat{U}(t)\,|p'\rangle\,\langle p'|\,|\psi,0\rangle = \int_{-\infty}^{-\infty} dp'\,|p'\rangle\,e^{-\frac{ip'^2}{2m\hbar}t}\psi(p',0)$$

Using this, we can then find the wavefunction in the momentum space, evolved with time.

$$\psi(p,t) = \langle p|\,|\psi\rangle$$
$$= \int_{-\infty}^{-\infty} dp'\,\langle p|\,|p'\rangle\,e^{-\frac{ip'^2}{2m\hbar}t}\psi(p',0) = e^{-\frac{ip^2}{2m\hbar}t}\psi(p,0)$$

To find the position space wavefunction, evolved with time, we must use the inverse Fourier transform.

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}px}\psi(p,t)dp$$

We then find the probability that the particle is in the region $0 \leq x \leq 1$ at time t.

$$P([0,1])(t) = \int_0^1 |\psi(x,t)|^2 dx$$

This process was performed with numeric integration in Mathematica. In order to avoid the cost of computing very large and very small numbers, $\hbar$ and m were expressed with time units of 10 kiloseconds. The results of the calculation for finding the particle in the 1 meter region are shown below.

Time	P(0; 1)
0	0.999959
1 minute	.947304
1 hour	.363355
1 day	.0158719
1 month	.000529105
1 year	.0000434881

Table 1: *Probabilities of finding the particle in the 1 meter region at various times*

These values were found with numerical integration.

From the data in Table 1, it is clear that by the point of 1 day, the particle is unlikely to be in the original region.

b. The probability was then plotted for various times on the order of kiloseconds.

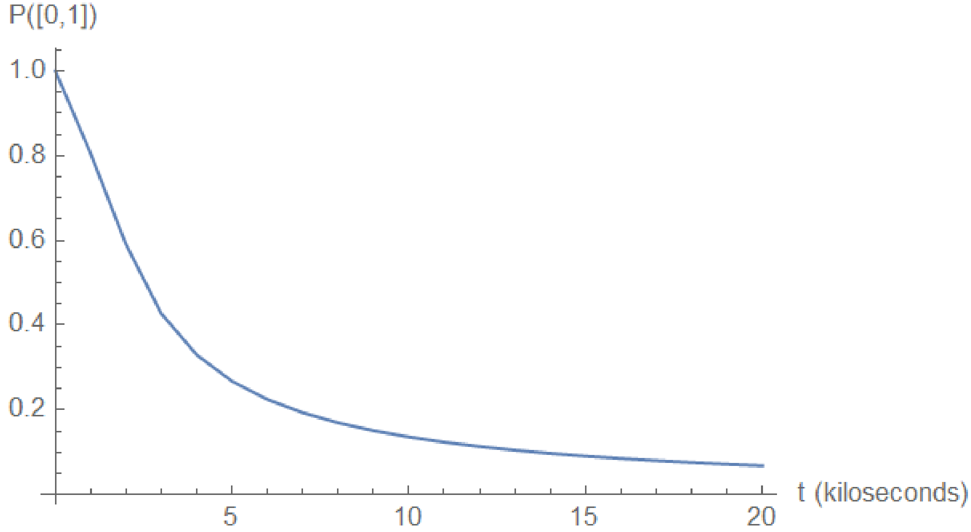
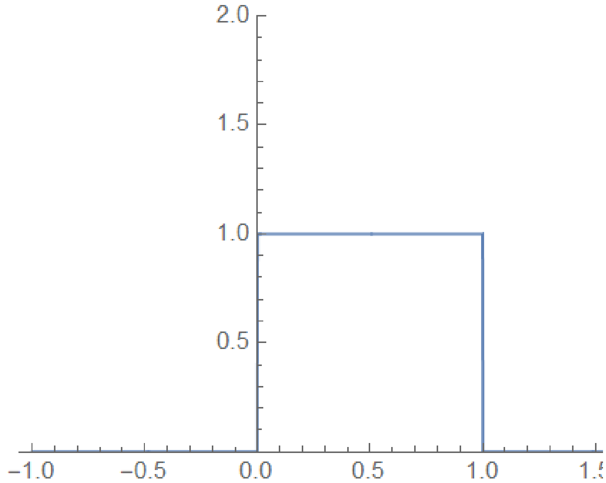


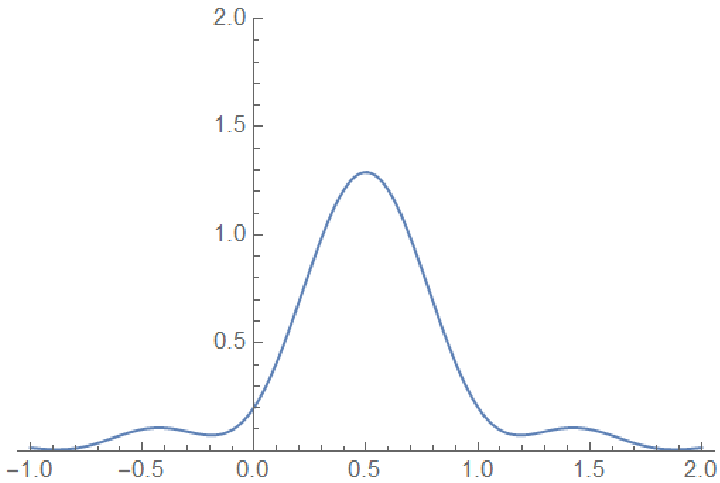
Figure 1: *Calculated probabilities for finding particle in initial 1 meter region with time*

It is not until time reaches the order of kiloseconds that the probability significantly decreases, and from there approaches zero. For reference, 1 kilosecond is about 17 minutes. The particle has a high probability of being in the region on the order of minutes, but the probability approaches zero on the order of hours. When time is very large, the probability of the particle being in the region approaches zero.

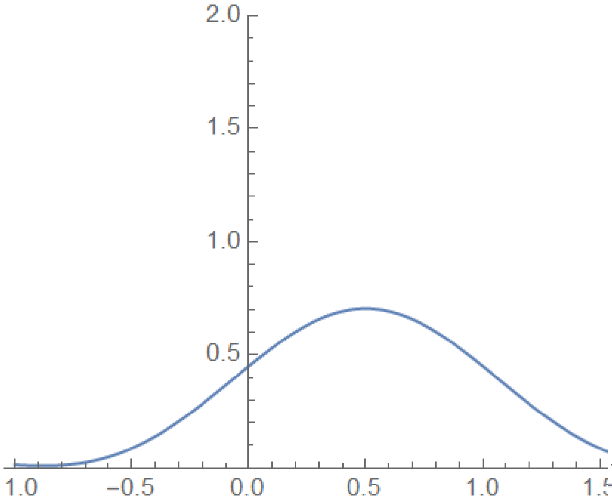
c. In order to get a more intuitive grasp on the behavior of this wavefunction, the probability amplitude was found around the initial region at various large times.



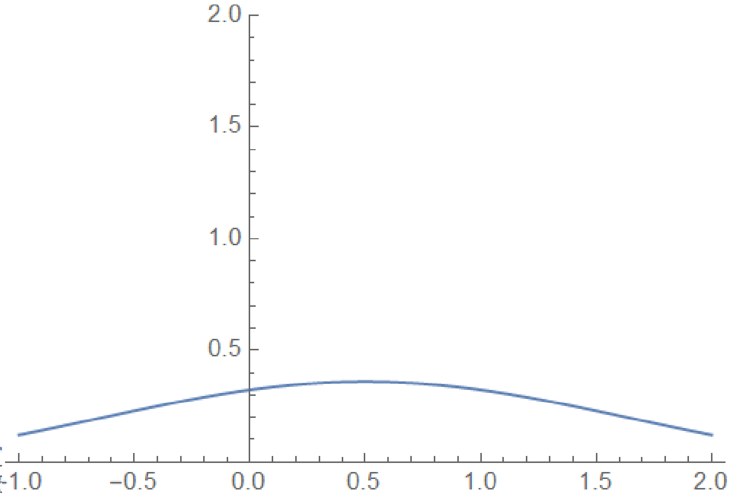
(a) Position probability distribution at t = 0



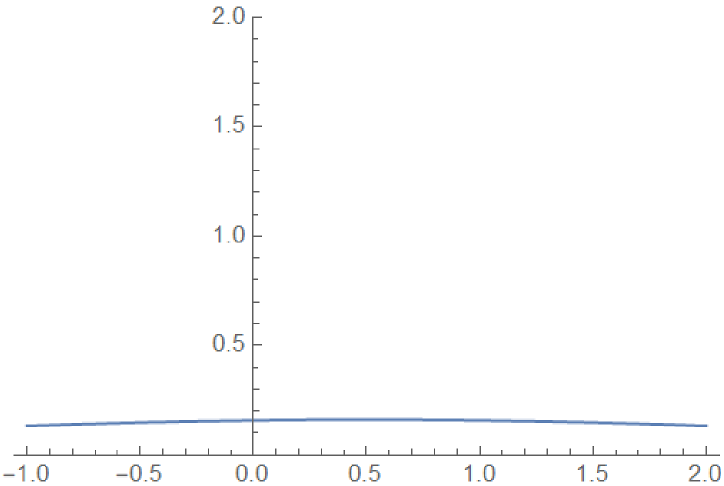
(b) Position probability distribution at t = 1 ks



(c) Position probability distribution at t = 2 ks



(d) Position probability distribution at t = 3 ks



(e) Position probability distribution at t = 10 ks

Figure 2: *Probability distribution of the particle around the initial region at times t = 0, 1 ks, 2 ks, 3 ks, and 10 ks.*

As can be seen in Fig. 2, the probability distribution is constant in the region $0 \leq$

$x \leq 1$ at time t = 0. Then, at t = 2 ks, the probability distribution increases in the center of the region. Thereafter, the distribution appears to take the shape of a Gaussian with a mean at .5m. This distribution widens and flattens until it approximates a flat line with zero amplitude as the time approaches infinity. At this point, the particle has equal probability of being found anywhere in space, and the only probability that satisfies this condition is zero.

We can compare this result to that of the Gaussian wave function as described in Problem 4. In this problem, we obtained the result:

$$\Delta X = \frac{a}{\sqrt{2}}\sqrt{1 + (\frac{t\hbar}{a^2 m})^2}$$

Let's consider the values of t required for the $\frac{t\hbar}{a^2 m}$ term to be on the order of one and therefore have a noticeable effect on ΔX.

Since the width of the initial function is 1m, a^2 is on the order of 1m. Therefore, t must be on the order of $\frac{\hbar}{m}$ in order for time evolution to have a significant effect on the wave-function. If we take $\hbar$ to be $6.58 * 10^{-22} MeVs$ and the mass of the electron to be $5.67 * 10^{-18}$MeV s^2 / m^2:

$$\frac{t\hbar}{m} \approx \frac{10^{-22}}{10^{-18}} t \approx 10^{-4} t$$

Therefore, t must be on the order of 10^4 seconds to have a noticeable effect on ΔX. This is consistent with the observed results for the constant amplitude case. Had a, the initial width, been smaller, the uncertainty in the momentum would have been larger and the wave-function would have spread more quickly with time.